Probabilistic Reachable Set Estimation for Uncertain Systems with Saturating Inputs

Carlo Karam[1], Matteo Tacchi-Bénard[1], Mirko Fiacchini[1]

[1] GIPSA-lab, Grenoble INP, Université Grenoble-Alpes, CNRS (France)

Introduction

Problem Setup and Stochastic Tubes

Given discrete-time LTI dynamics:

$$x_{k+1} = Ax_k + Bu_k + w_k$$

Noise Characteristics

Unbounded, i.i.d., with:

$$\mathbb{E}[w_k] = 0, \quad \mathbb{E}[w_k^\top w_k] = W.$$

Constraints

$$\Pr\{x \in \mathcal{X}\} \geq 1 - \varepsilon_x,$$

$$\Pr\{u \in \mathcal{U}\} \geq 1 - \varepsilon_u.$$

- Parameterize $u_k = v_k + Ke_k$, decompose $x_k = z_k + e_k$, such that:

MPC action

$$z_{k+1} = Az_k + Bv_k, \qquad e_{k+1} = (A + BK)e_k + w_k,$$

- Impose $z \in \mathcal{X} \ominus \mathcal{R}_e$ and $v \in \mathcal{U} \ominus K\mathcal{R}_e$.

probabilistic reachable set

Problem Setup and Stochastic Tubes

Given discrete-time LTI dynamics:

$$x_{k+1} = Ax_k + Bu_k + w_k$$

Noise Characteristics

Unbounded, i.i.d., with:

$$\mathbb{E}[w_k] = 0, \quad \mathbb{E}[w_k^\top w_k] = W.$$

Constraints

$$\Pr\{x \in \mathcal{X}\} \geq 1 - \varepsilon_x,$$

$$\Pr\{u \in \mathcal{U}\} \geq 1 - \varepsilon_u.$$

- Parameterize $u_k = \underbrace{v_k}_{\text{MPC action}} + Ke_k$, decompose $x_k = z_k + e_k$, such that:

$$z_{k+1} = Az_k + Bv_k, \qquad e_{k+1} = (A + BK)e_k + w_k,$$

- Impose $z \in \mathcal{X} \ominus \mathcal{R}_e$ and $v \in \mathcal{U} \ominus K\mathcal{R}_e$.

Actuator saturation occurs for any $u \notin \mathcal{U}$!

Objectives

Our goal is to maintain the stochastic tubes approach for its efficient closed-loop implementation while:

1. Guaranteeing state chance constraints (deterministic under some tightening, i.e. $z \in \mathcal{X} \ominus \mathcal{R}_e$).

2. Enforcing hard input constraints in the presence of **unbounded noise**.

3. Minimizing the conservatism of $\mathcal{R}_e$.

Main Focus

We need a suitable synthesis of Probabilistic Reachable Sets $\mathcal{R}_e$.

Methodology

Our Proposal

- Hard input constraints can be enforced by saturating the control:

$$x_{k+1} = Ax_k + B\underbrace{\varphi}_{\text{component-wise saturation function}}(u_k) + w_k$$

- Following the same stochastic tubes approach, we get:

$$z_{k+1} = \underbrace{A}_{\text{Open-loop stable}} z_k + Bv_k \qquad e_{k+1} = \underbrace{Ae_k + B(\varphi(v_k + Ke_k) - v_k)}_{f(e_k, v_k)} + w_k$$

$f(e_k, v_k)$ is nonlinear and dependent on v. How do we construct PRS?

Main Idea

Construct a linear bound on $f(e_k, v_k)$ valid for any admissible v.

Handling Saturations

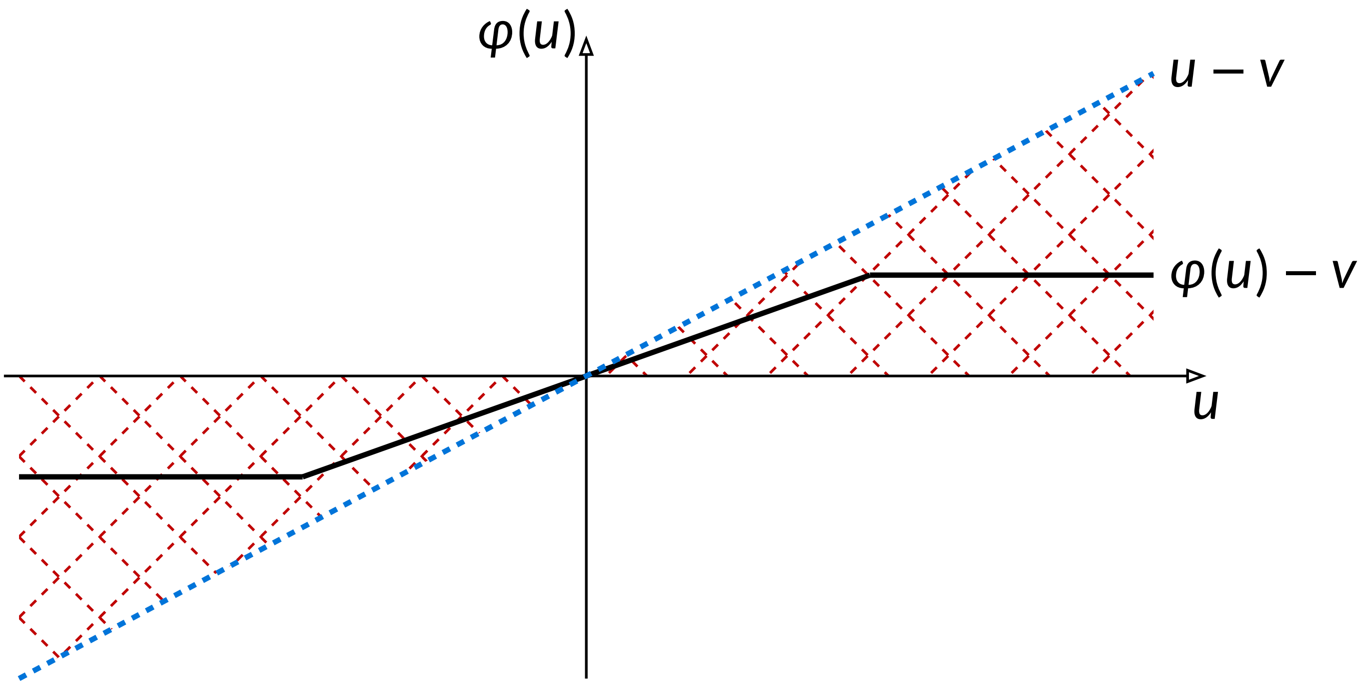
1. Contain v within saturation bounds, $v \in \mathcal{V} \subseteq \mathcal{U}$.

Handling Saturations

1. Contain v within saturation bounds, $v \in \mathcal{V} \subseteq \mathcal{U}$.

2. Linearly bound $\varphi(Ke+v)-v$ within a **sector**.

$$\begin{cases} 0 \leq \varphi(K_i e + v_i) - v_i \leq K_i e & \text{if } K_i e \geq 0 \\ K_i e \leq \varphi(K_i e + v_i) - v_i \leq 0 & \text{if } K_i e < 0 \end{cases}$$



Handling Saturations

1. Contain v within saturation bounds, $v \in \mathcal{V} \subseteq \mathcal{U}$.

2. Linearly bound $\varphi(Ke+v)-v$ within a **sector**.

$$\begin{cases} 0 \leq \varphi(K_i e + v_i) - v_i \leq K_i e & \text{if } K_i e \geq 0 \\ K_i e \leq \varphi(K_i e + v_i) - v_i \leq 0 & \text{if } K_i e < 0 \end{cases}$$

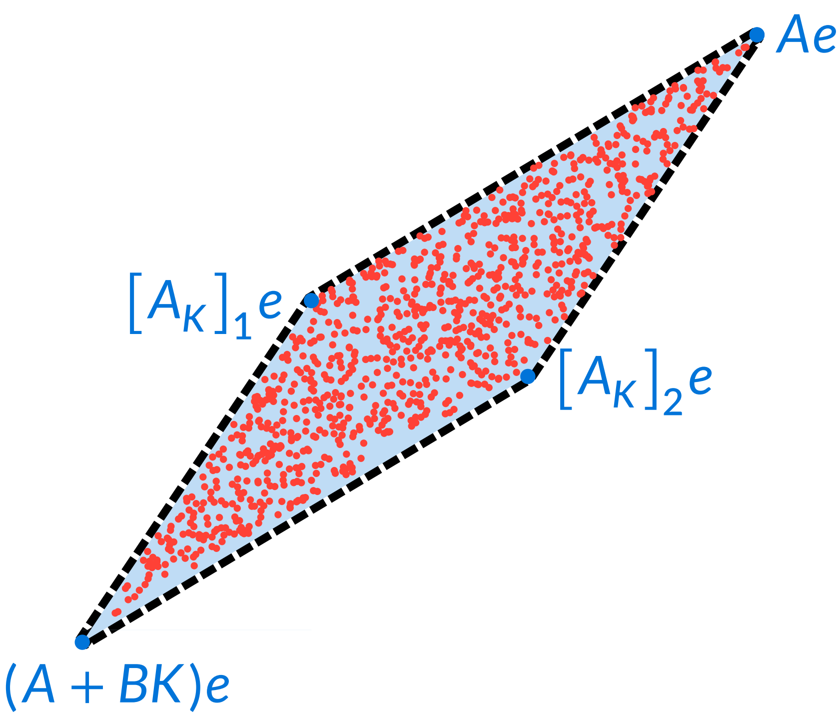
3. Replace the nonlinearity with a convex representation.

$$\boxed{f(e,v) \in \mathrm{co}\left\{ \underbrace{\left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i \right)}_{[A_K]_i} e \right\}_{\mathcal{J} \subseteq \mathbb{N}^m}}$$

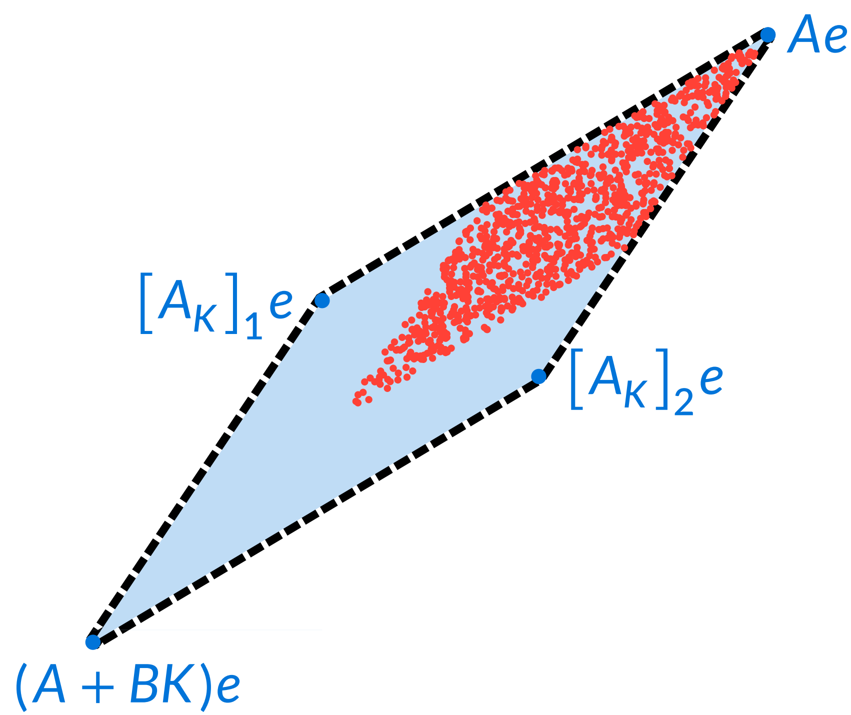
$f(e,v)$ is bounded from all sides by a convex polytope encoding different saturation scenarios!

A Visual Representation of the Convex Bound

For some e, and any $v \in \mathcal{V}$, the bound holds with varying degree of conservatism. For $m = 2$:



$\mathcal{V} = \mathcal{U}$



$\mathcal{V} \subset \mathcal{U}$

Contracting Dynamics

1

Linear System Contraction (open-loop)

$$\left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right)^{\top} P \left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right) \preceq \lambda P \qquad \mathcal{J} \subseteq \mathbb{N}^m$$

Design Parameters, $P \succ 0$, $\lambda \in [0, 1)$

Contracting Dynamics

1

via bound

2

Linear System Contraction (open-loop)

$$\left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right)^{\top} P \left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right) \preceq \lambda P \qquad \mathcal{J} \subseteq \mathbb{N}^m$$

Design Parameters, $P \succ 0$, $\lambda \in [0, 1)$

Nonlinear System Contraction

$$f(e, v)^{\top} P f(e, v) \preceq \lambda P$$

Contracting Dynamics

1 via bound → 2 incorporate noise → 3

Linear System Contraction (open-loop)

$$\left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right)^\top P \left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right) \preceq \lambda P \qquad \mathcal{J} \subseteq \mathbb{N}^m$$

Design Parameters, $P \succ 0$, $\lambda \in [0, 1)$

Nonlinear System Contraction

$$f(e, v)^\top P f(e, v) \preceq \lambda P$$

Expected Error Contraction

$$\mathbb{E}[e_{k+1}^\top P e_{k+1}] \leq \underbrace{\lambda \mathbb{E}[e_k^\top P e_k]}_{\text{Contraction Effect}} + \underbrace{\operatorname{tr}(PW)}_{\text{Noise Effect}}$$

Constructing Reachable Sets

4

Closed-form Bound on Expectation

$$\mathbb{E}[e_k^\top P e_k] \leq \frac{1-\lambda^k}{1-\lambda}\,\mathrm{tr}(PW) \leq \frac{1}{1-\lambda}\,\mathrm{tr}(PW)$$

Constructing Reachable Sets

4

Markov inequality

5

Closed-form Bound on Expectation

$$\mathbb{E}[e_k^\top P e_k] \leq \frac{1-\lambda^k}{1-\lambda}\,\mathrm{tr}(PW) \leq \frac{1}{1-\lambda}\,\mathrm{tr}(PW)$$

Probabilistic Reachable Set (PRS)

$$\mathcal{R}_k^{\varepsilon_x}(\lambda) : \left\{ e \;\middle|\; e^\top P e \leq \frac{1-\lambda^k}{\varepsilon_x(1-\lambda)}\,\mathrm{tr}(PW) \right\}$$

Constructing Reachable Sets

4

Markov inequality

5

taking lim as $k \to \infty$

6

Closed-form Bound on Expectation

$$\mathbb{E}[e_k^\top P e_k] \leq \frac{1-\lambda^k}{1-\lambda}\,\mathrm{tr}(PW) \leq \frac{1}{1-\lambda}\,\mathrm{tr}(PW)$$

Probabilistic Reachable Set (PRS)

$$\mathcal{R}_k^{\varepsilon_x}(\lambda) : \left\{ e \;\middle|\; e^\top P e \leq \frac{1-\lambda^k}{\varepsilon_x(1-\lambda)}\,\mathrm{tr}(PW) \right\}$$

Probabilistic Ultimate Bound (PUB)

$$\mathcal{R}_\infty^{\varepsilon_x}(\lambda) : \left\{ e \;\middle|\; e^\top P e \leq \frac{1}{\varepsilon_x(1-\lambda)}\,\mathrm{tr}(PW) \right\}$$

Reducing Conservatism

Open-loop Contraction

- λ is a conservative contraction rate leading to very loose bounds.

Open-loop Contraction

- λ is a conservative contraction rate leading to very loose bounds.

Intuition

$$\left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right)^\top P \left(A + \sum_{i \in \mathcal{J}} B_{(i)} K_i\right) \preceq \lambda P$$

is equivalent to:

$$A^\top P A \preceq \lambda P \qquad \text{Limiting Constraint}$$

$$\left(A + B_{(1)} K_1\right)^\top P \left(A + B_{(1)} K_1\right) \preceq \lambda P$$

$$\vdots$$

$$\left(A + B_{(1:2)} K_{1:2}\right)^\top P \left(A + B_{(1:2)} K_{1:2}\right) \preceq \lambda P$$

$$\vdots$$

$$(A + BK)^\top P (A + BK) \preceq \lambda P$$

Open-loop Contraction

- λ is a conservative contraction rate leading to very loose bounds.
  $\Rightarrow$ sets don't take into account the stabilizing effect of feedback.

We can do better!

When $\varphi(u) = u$, a stronger contraction $\lambda_L \leq \lambda$ holds for the already computed K, P.

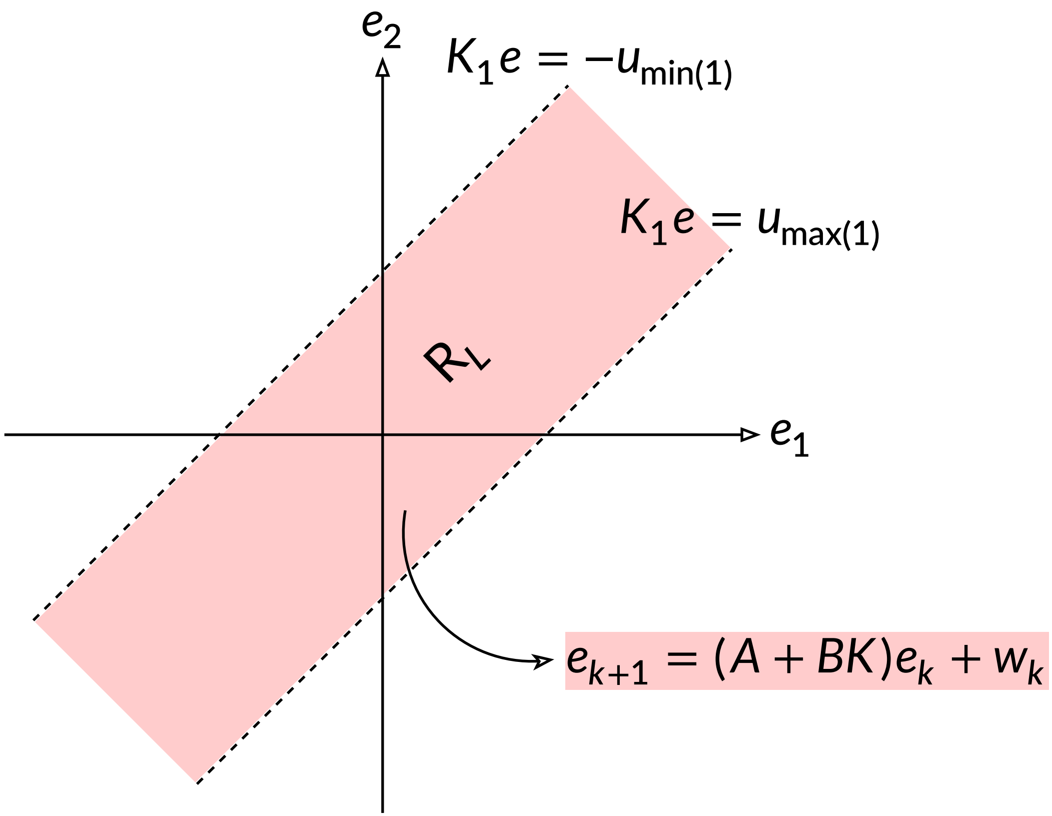
$$(A + BK)^\top P(A + BK) \preceq \lambda_L P$$

$$\mathbb{E}\left[e_{k+1}^\top P e_{k+1}\right] \leq \lambda_L \mathbb{E}\left[e_k^\top P e_k\right] + \mathrm{tr}(PW)$$

How can we account for this contraction in our analysis?

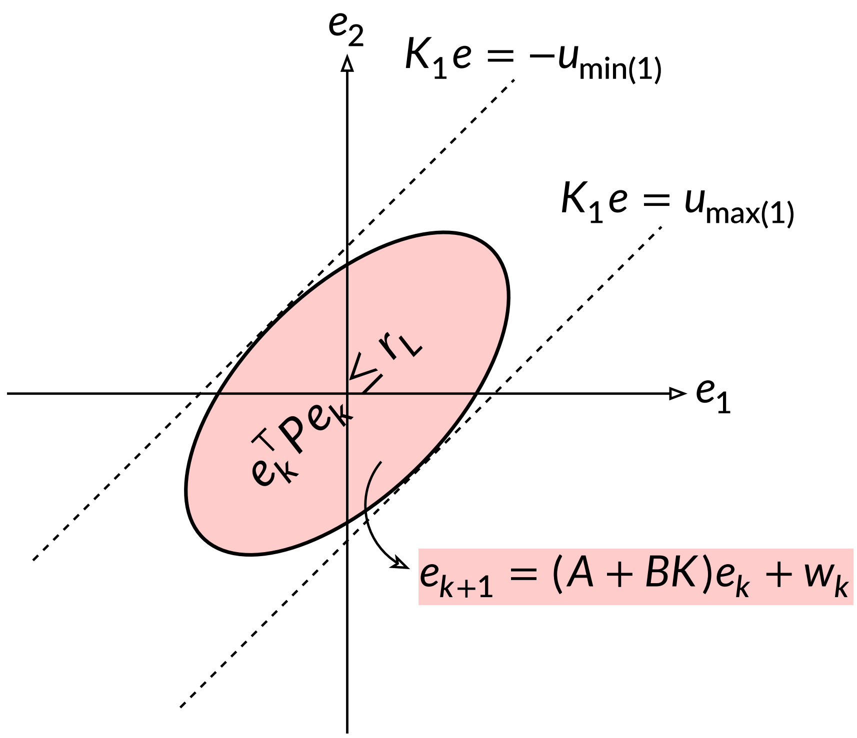
Saturation Region of Linearity

The system does not saturate ($\varphi(u) = u$) when we are in its region of linearity. R_L is the true polyehdral region of linearity. It may be unbounded in certain (unconstrained) directions.



Saturation Region of Linearity

The system does not saturate ($\varphi(u) = u$) when we are in its region of linearity. R_L is the true polyehdral region of linearity. It may be unbounded in certain (unconstrained) directions.



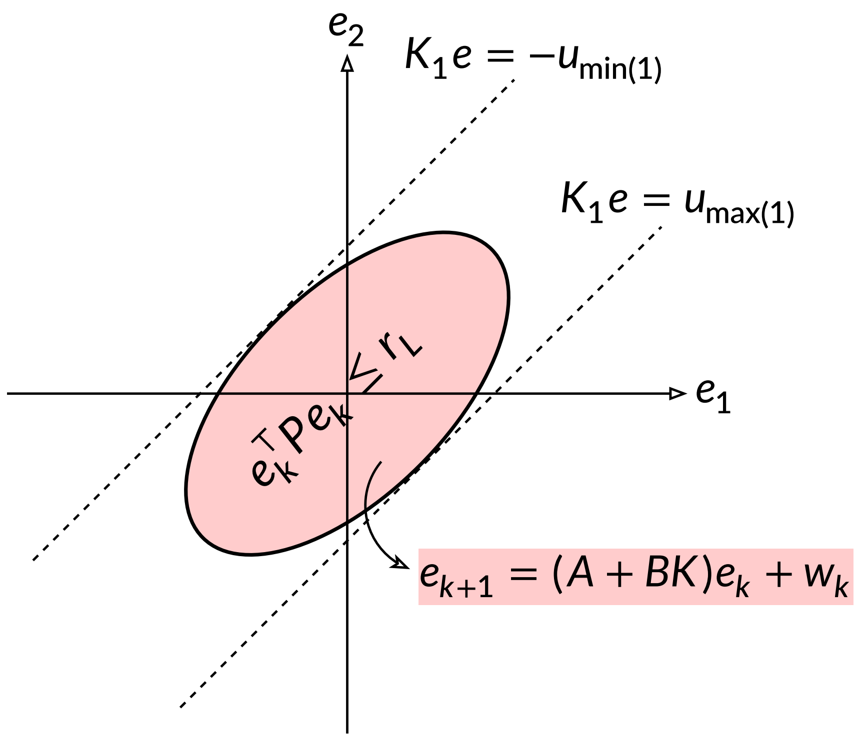
Ellipsoidal Region of Linearity

R_L can be under-approximated by an ellipsoid with:

- shape matrix P.
- scaling r_L, that grows with the energy allocated to the feedback Ke.

Saturation Region of Linearity

The system does not saturate ($\varphi(u) = u$) when we are in its region of linearity. R_L is the true polyehdral region of linearity. It may be unbounded in certain (unconstrained) directions.



Ellipsoidal Region of Linearity

R_L can be under-approximated by an ellipsoid with:

- shape matrix P.
- scaling r_L, that grows with the energy allocated to the feedback Ke.

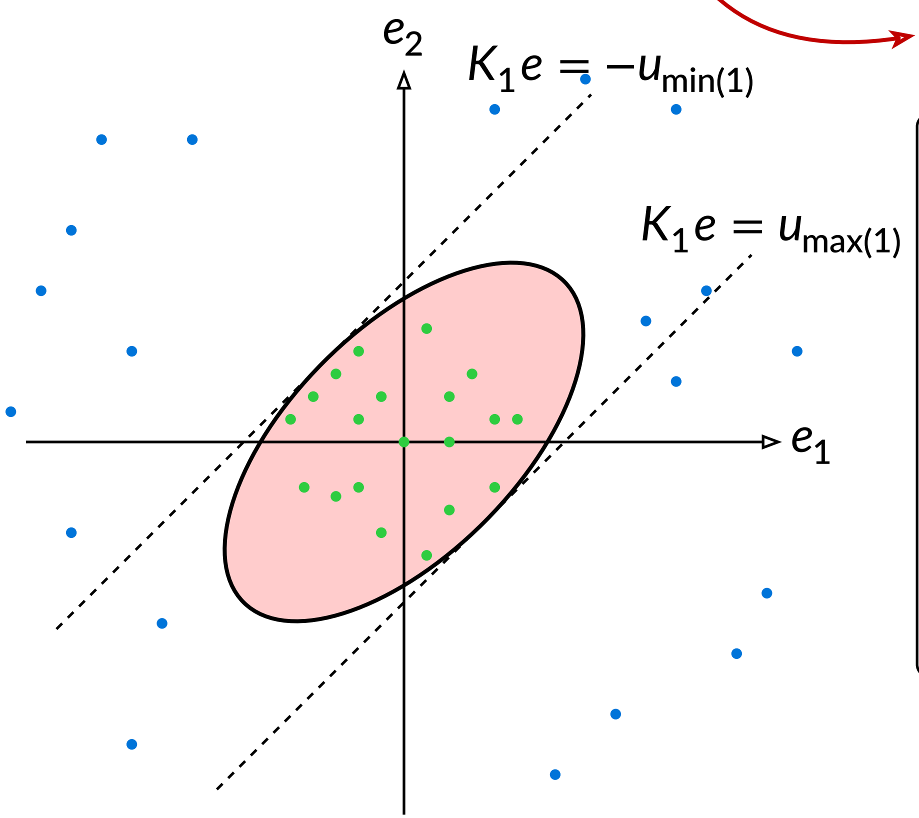
How can we integrate this into our bound?

Weighted Bound on the Error

Main Idea

The system probabilistically switches between **saturated** and **unsaturated** modes.

Can we characterize this?



Use Markov inequality:

$$\Pr\{e_k^\top P e_k \le r_L\} \ge \boxed{1 - \frac{E[e_k^\top P e_k]}{r_L}}$$

Informative if r_L is "large enough"

$$\Pr\{e_k^\top P e_k > r_L\} = 1 - \Pr\{e_k^\top P e_k \le r_L\}$$

Weighted Bound on the Error

We take into account both saturated and unsaturated contractive behaviors by weighing their contributions using $\Pr\{e_k^\top P e_k \leq r_L\}$.

Effective Contraction Rate

If the ellipsoidal region of linearity has a "large enough" scaling

$$r_L \geq \boxed{\frac{1}{1-\lambda}\operatorname{tr}(PW),}$$

$\lim_{k\to\infty}$ of $\mathbb{E}[e_k^\top P e_k]$,
"strong" condition

then,

$$\mathbb{E}\left[e_{k+1}^\top P e_{k+1}\right] \leq \bar{\lambda}\,\mathbb{E}\left[e_k^\top P e_k\right] + \operatorname{tr}(PW),$$

for all $k \in \mathbb{N}$ where $\bar{\lambda}$ satisfies,

$$\frac{1}{1-\bar{\lambda}}\operatorname{tr}(PW) = \frac{\bar{\lambda}-\lambda_L}{\lambda-\lambda_L} r_L.$$

Weighted Bound on the Error

We take into account both saturated and unsaturated contractive behaviors by weighing their contributions using $\Pr\{e_k^\top P e_k \leq r_L\}$.

Effective Contraction Rate

If the ellipsoidal region of linearity has a "large enough" scaling

$$r_L \geq \frac{1}{1-\lambda}\operatorname{tr}(PW),$$

then,

$$\mathbb{E}\left[e_{k+1}^\top P e_{k+1}\right] \leq \bar{\lambda}\,\mathbb{E}\left[e_k^\top P e_k\right] + \operatorname{tr}(PW),$$

⇒ **compute PRS, PUB**

for all $k \in \mathbb{N}$ where $\bar{\lambda}$ satisfies,

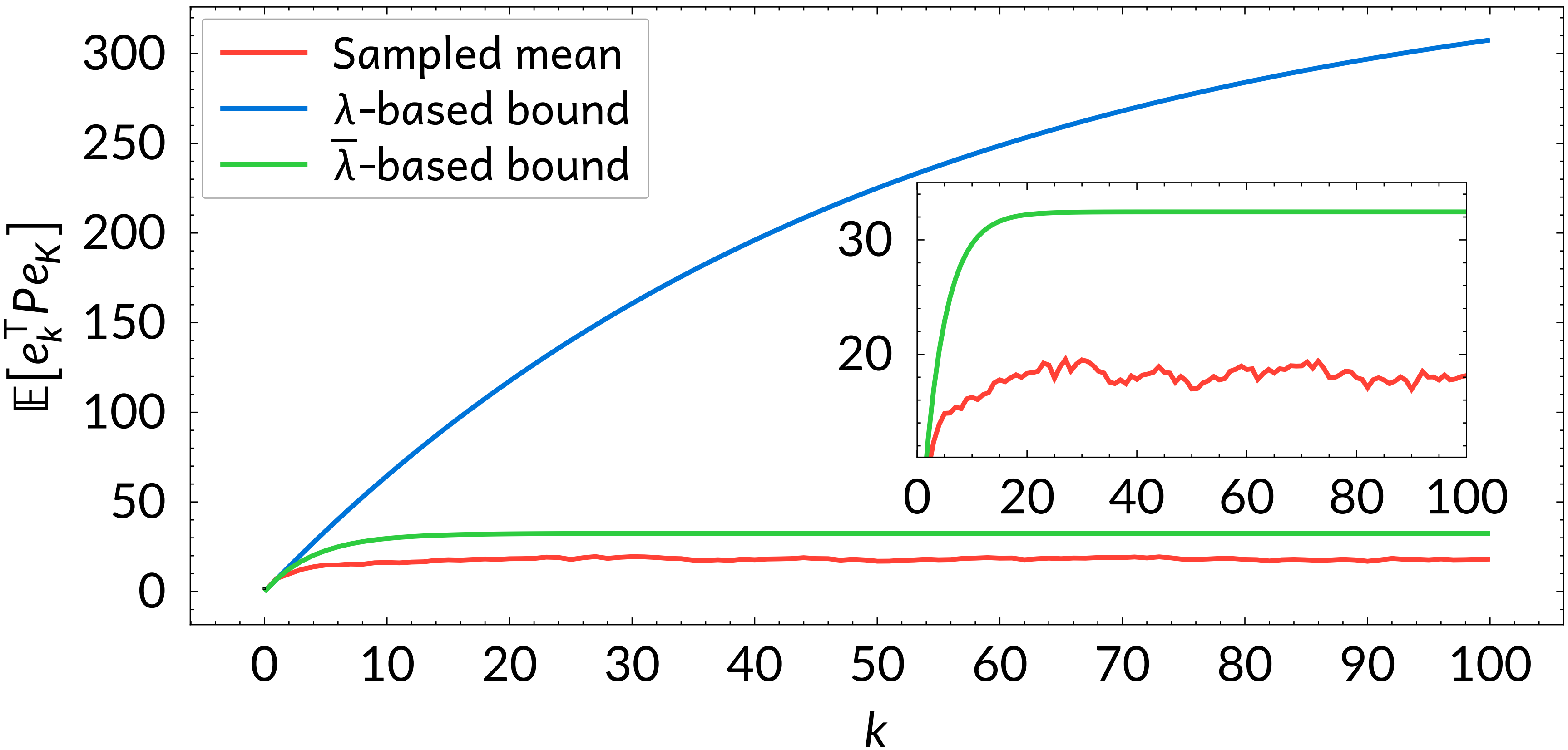
$$\frac{1}{1-\bar{\lambda}}\operatorname{tr}(PW) = \frac{\bar{\lambda}-\lambda_L}{\lambda-\lambda_L} r_L.$$

⇒ **quadratic equation with 1 admissible solution**

Tightness of the New Bound

The improvement can be first noticed on the tightness of the expectation bound.

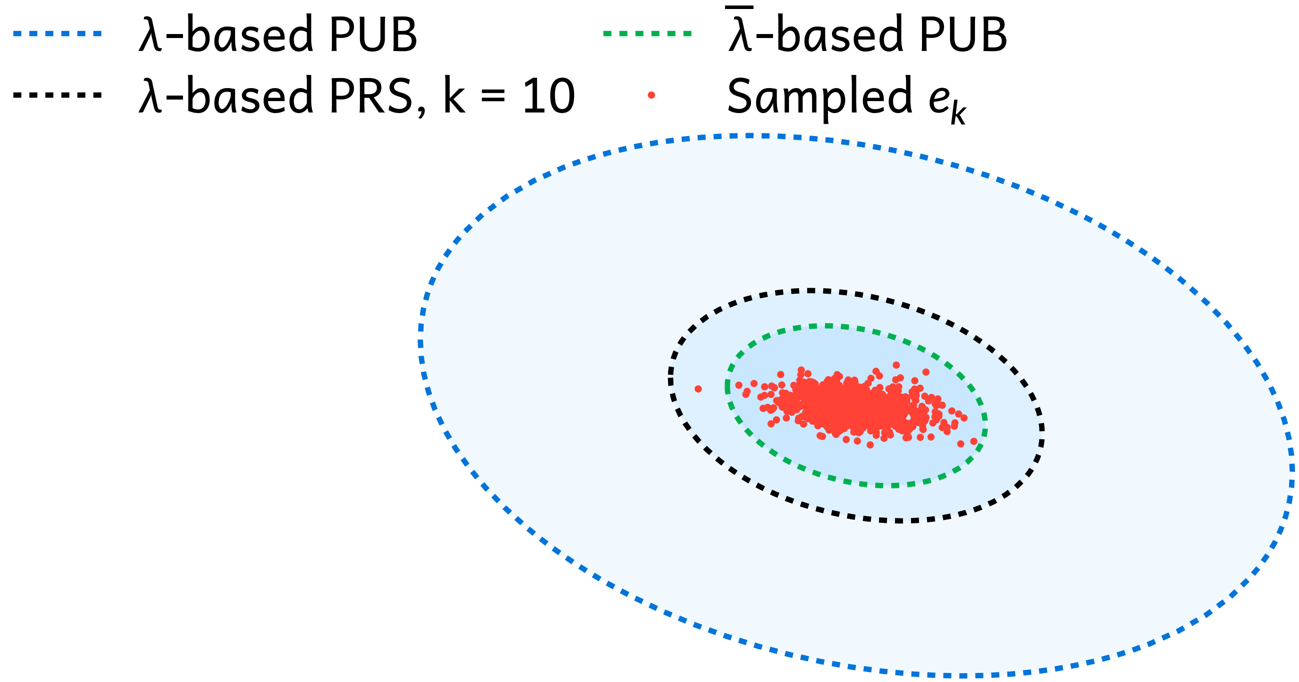
$\lambda = 0.9802,\ \bar{\lambda} = 0.7826,\ \varepsilon_x = 0.2, W = \mathrm{I}_2.$



Tightness of the New Bound

Equivalently, the sets themselves are much tighter. Comparing the **effective PUB** to the **open-loop PUB**.

$\lambda = 0.9802$, $\bar{\lambda} = 0.7826$, $\varepsilon_x = 0.2$, $W = I_2$.



Summary and Outlook

Retracing our steps:

1. We **decouple** the problem through the **generalized sector bound**.
2. We **linearize** the problem through the **polytopic inclusion**.
3. We compute always valid open-loop PRS and PUB using standard methods.
4. We compute tighter PRS and PUB **incorporating linear behavior** under favorable conditions.

Looking ahead:

- Integration into SMPC scheme.
- Further reduction of conservatism using the Moment Sum-of-Squares hierarchy to compute semi-algebraic sets.

Thank You!